The limits of leadership advantage in Stackelberg duopoly

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Abstract

This paper characterizes the worst-case first-mover advantage and surplus sharing limits for a Stackelberg duopoly facing α-concave demand. Under constant and symmetric marginal costs, I prove that across all possible equilibria, the leader-to-follower profit ratio is at least $\frac{1+\sqrt{5+4\alpha}}{2}$. Thus, the leader is guaranteed to earn at least 61.8% more than the follower under log-concave demand, and at least twice as much under concave demand, though the relative advantage itself can be arbitrarily large. Furthermore, I derive a lower bound for the leader's profit relative to the efficient surplus, alongside an upper bound for the deadweight loss. Extremal demand curves D for which D^α and $\ln(D)$ are affine attain all bounds.

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1 Introduction

In a Stackelberg duopoly, the strategic value of leadership depends fundamentally on the geometry of demand (e.g., von Stackelberg, 1934; Gal-Or, 1985; Dowrick, 1986). While economists routinely rely on convenient functional forms to quantify the first-mover advantage, such parametric assumptions can overstate precision and obscure broader theoretical limits. For industrial organization and antitrust policy, evaluating market power often does not require predicting equilibrium under an idealized demand specification. Instead, the primary task is establishing absolute, worst-case guarantees for surplus division and efficiency loss when the true shape of demand is unknown.

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This paper determines these limits for sequential competition through the lens of α-concave demand. Introduced by Caplin and Nalebuff (1991a,b), this restriction requires $D(p)^\alpha$ to be concave for any positive α, with log-concavity as a special case when $\alpha = 0$. The class can capture a broad spectrum of familiar economic demand functions (An, 1998; Bagnoli and Bergstrom, 2005). While prior work uses α-concavity to establish profit floors under pure monopoly (Condorelli, 2022; Gui and Huang, 2022; Le, 2024), this paper is the first to show how it restricts the range of first-mover advantage.

My first contribution is a distribution-free lower bound on the leadership advantage. I show that, at *any* Stackelberg equilibrium, regardless of multiplicity, the follower's share of industry profit equals the price elasticity of demand. Because the leader's optimality condition and α-concavity together impose a tight upper bound on all possible equilibrium elasticities, the leader-to-follower profit ratio cannot fall below $\frac{1+\sqrt{5+4\alpha}}{2}$. Remarkably, it reduces to the Golden Ratio (≈ 1.618) for the vast class of log-concave demands and to the classic benchmark of 2 when demand is concave. Conversely, I demonstrate by example that this relative advantage has no upper bound.

My second contribution are bounds on the leader's profit and the resulting distortion. I show that the leader always secures a definitive fraction of the potential surplus, while deadweight loss is strictly capped relative to industry profit. This yields a striking welfare conclusion: introducing just one follower substantially compresses the worst-case welfare losses of a monopoly across the entire α-concave demand class. With log-concavity, the maximum deadweight-loss-to-profit ratio falls from $e - 2 \approx 71.8\%$ under pure monopoly (Le, 2024) to 21.8% under Stackelberg competition. With standard concavity, the corresponding bound drops from 1/2 to 1/6. From a regulation standpoint, this suggests that promoting even marginal market entry is favorable for market efficiency. Finally, by using generalized exponential demands, I show that all the derived bounds are sharp.

Related Literature This paper advances a growing literature on robust surplus division under imperfect competition. At the most general level, Tamuz (2013) establishes a universal lower bound on monopoly profit relative to the geometric mean of consumer valuations. By imposing log-concavity or α-concavity on the demand, Condorelli (2022) and Le (2024) tighten these bounds relative to the arithmetic mean, a clearer measure that captures the entire economic pie. Complementing α-concavity, Ge (2025) extends Myerson's (1981) regularity condition and then derives bounds on monopoly profit under α-regular demand. Shifting to strategic markets, Condorelli and Szentes (2022) study surplus sharing limits in Cournot oligopoly. Beyond pure monopoly and oligopoly, Kang and Vasserman (2025) provide tools to bound how sensitive policy welfare effects are to underlying demand assumptions, while von Beringe and Whitmeyer (2025) analyze robust welfare in imperfect competition under both demand and supply ambiguity.

2 Stackelberg competition

Consider a homogeneous-goods market supplied by a sequential duopoly, where a leader and a follower compete in quantity with a common constant marginal cost. Let the demand curve be represented by a continuous, strictly decreasing function $D : [0, \omega] \to [0, 1]$.[1] We normalize $D(0) = 1$ and $D(\omega) = 0$, and assume that the area under demand,

$$\mu \triangleq \int_0^\omega D(p)\, dp,$$

is finite. The value μ represents the maximum potential surplus to be shared, whereas $\omega > 0$ (inclusive of $\omega = \infty$) denotes the maximum willingness to pay.[2] Normalizing marginal cost to zero is without loss of generality. In fact, for any positive marginal cost c, the analysis holds verbatim by using instead the truncated demand $\tilde{D}(p) \triangleq D(p + c)$. I further assume demand is α-*concave* for a fixed $\alpha \in [0, 1]$. By definition, this requires $D(p)^\alpha$ to be concave for $\alpha \in (0, 1]$, and $\ln D(p)$ to be concave for $\alpha = 0$. A key property is nesting: α_1-concavity implies α_2-concavity whenever $\alpha_1 \geq \alpha_2$.

To solve for a subgame-perfect equilibrium, I assume that D is twice continuously differentiable on $[0, \omega]$.[3] Let $P = D^{-1}$ denote the inverse demand function. The leader moves first, choosing quantity $q_L \in [0, 1]$. After observing q_L, the follower chooses quantity $q_F \in [0, 1 - q_L]$, resulting in a market clearing price $p = P(q_L + q_F) \in [0, \omega]$. The follower maximizes their profit $\Pi_F = P(q_L + q_F)q_F$. For every given $q_L \in [0, 1)$, there is a unique interior solution characterized by the first-order condition $P(Q) + q_F P'(Q) = 0$, where $Q \triangleq q_L + q_F$.[4] Because $Q = D(p)$ and $D'(p) = 1/P'(Q)$, it is convenient to parameterize the outputs for both firms by market price p. Substituting these identities into the first-order condition yields the follower's best-response quantity:

$$q_F(p) = -pD'(p). \tag{1}$$

By market clearing, $q_L + q_F = D(p)$, which leaves the leader with the effective demand:

$$q_L(p) = D(p) + pD'(p). \tag{2}$$

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[1] From a probabilistic perspective, $F(p) = 1 - D(p)$ is the cumulative distribution of consumer valuations.

[2] Since $\mu \leq \omega$ and μ is the expected willingness to pay, a finite ω ensures finite μ, rendering the assumption of finite μ unnecessary. When $\omega = \infty$, the assumption of finite μ is necessary. When $\omega = \infty$, the notation $[0, \omega)$ and $[0, \omega]$ both represent the same set, $[0, \infty)$ and $D(\omega) = 0$ means $\lim_{p\to\infty} D(p) = 0$.

[3] While log-concavity implies differentiability almost everywhere (Condorelli, 2022; Le, 2024), the sequential nature of Stackelberg competition requires a stronger assumption. Because the leader optimizes anticipating the follower's best response, it necessitates a twice-differentiable demand function.

[4] For any leader quantity $q_L \in [0, 1)$, the follower chooses $p \in [0, P(q_L)]$ to maximize $\Pi_F(p; q_L) = p(D(p) - q_L)$. Because profit vanishes at the boundaries ($p = 0$ and $p = P(q_L)$), the strict concavity of the log-profit function, $\ln p + \ln(D(p) - q_L)$, guarantees a unique interior maximum. The resulting first-order condition alternatively establishes Equations (1) and (2).

These output shares can thus be expressed succinctly using the price elasticity of demand, $\epsilon_p \triangleq -pD'(p)/D(p)$. This substitution yields $q_F(p) = \epsilon_p D(p)$ and $q_L(p) = (1-\epsilon_p)D(p)$. This reveals that the division of output (and industry profit) is governed by the elasticity: the follower captures ϵ_p of the market share, leaving $1-\epsilon_p$ for the leader.[5]

Anticipating the follower's strategic response, the leader selects an induced price $p \in [0,\omega]$ to maximize the *continuation profit function*:

$$\Pi_L(p) = p[D(p) + pD'(p)] = pD(p)(1-\epsilon_p). \tag{3}$$

Because α-concavity implies log-concavity, the hazard rate $h(p) \triangleq -D'(p)/D(p)$ is increasing. This implies that the price elasticity of demand, $\epsilon_p = p \cdot h(p)$, strictly increases on $[0,\omega)$. Elasticity spans from $\epsilon_0 = 0$ to an unbounded limit as price approaches ω, regardless of whether ω is finite or infinite. By strict monotonicity and the Intermediate Value Theorem, there is a *unique* price $p^M \in (0,\omega)$ such that $\epsilon_{p^M} = 1$, which is precisely the single-firm *monopoly price*. Therefore, the leader's effective pricing domain is $[0,p^M]$. This corresponds the leader's possible quantity choices from full market saturation ($q_L = 1$ inducing $p = 0$) to complete market exit ($q_L = 0$ inducing $p = p^M$).

3 The limits of leadership advantage

A worst-case leadership advantage is only meaningful if an equilibrium always exists. Fortunately, α-concave demand guarantees the existence of an equilibrium. While this equilibrium need not be unique, every Stackelberg equilibrium price is interior and satisfies the necessary first-order condition.

Lemma 1 (Existence and Interiority of Equilibria)**.** *A global maximum for the leader's continuation profit exists and occurs at an interior price* $p^S \in (0,p^M)$. *Moreover, any Stackelberg equilibrium price* p^S *satisfies* $D(p^S) + 3p^S D'(p^S) + (p^S)^2 D''(p^S) = 0$.

Proof. The leader's profit can be written as $\Pi_L(p) = pD(p)(1-\epsilon_p)$. Because $pD(p) > 0$ for all $p \in (0,\omega)$, the profit is strictly positive if and only if $p \in (0,p^M)$, and zero at the boundaries $p \in \{0,p^M\}$. Any price $p > p^M$ yields strictly negative profit. Therefore, the leader's effective optimization domain is restricted to the compact set $[0,p^M]$. Because $D(p)$ is assumed to be twice continuously differentiable, $\Pi_L(p)$ is a continuous function on the compact set $[0,p^M]$. By the Extreme Value Theorem, a global maximum exists. Also, because $\Pi_L(0) = 0$, $\Pi_L(p^M) = 0$, and $\Pi'_L(0) = D(0) = 1$, the maximum must occur at an interior Stackelberg price $p^S \in (0,p^M)$. Along with the differentiability of $\Pi_L(p)$, any global maximizer p^S satisfies the first-order condition $\Pi'_L(p^S) = 0$. This implies $D(p^S) + 3p^S D'(p^S) + (p^S)^2 D''(p^S) = 0$ as desired. □

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[5] If the leader chooses $q_L = 1$, the price is driven to $p = 0$ and the follower is forced to produce $q_F = 0$. In this corner case, the demand elasticity is $\epsilon_0 = 0$ and the market share identity holds trivially.

Remark 1 (Leader-to-Follower Profit Ratio)**.** Let $\epsilon^S \triangleq \epsilon_{p^S}$ denote an equilibrium elasticity of demand. Because any Stackelberg price $p^S \in (0, p^M)$ and ϵ_p is strictly increasing with $\epsilon_0 = 0$ and $\epsilon_{p^M} = 1$, it follows that $\epsilon^S \in (0, 1)$. Using the share identities from Equation (1) and Equation (2), the equilibrium leader-to-follower profit ratio is:

$$\frac{\Pi_L(p^S)}{\Pi_F(p^S)} = \frac{p^S q_L(p^S)}{p^S q_F(p^S)} = \frac{1-\epsilon^S}{\epsilon^S}. \tag{4}$$

While α-concavity guarantees a unique best response for the follower, it does not ensure a unique strategy for the leader.

Example 1 (Multiplicity of Equilibria)**.** Consider the following *hyperbolic demand*:

$$D(p) = \exp\left(-\left[ap + b\sqrt{c + (p-p_0)^2} - b\sqrt{c + p_0^2}\right]\right), \quad p \in [0, \infty).$$

By construction, $D(0) = \exp(0) = 1$ and $D(\infty) = 0$. Provided $a, b, c > 0$, the exponent is concave, ensuring demand is log-concave. However, calibrating this demand with $a = 0.2$, $b = 0.2$, $c = 0.01$, and $p_0 \approx 0.598$ generates two global maxima. The Stackelberg leader is indifferent between a high-quantity strategy ($p^S \approx 0.590$) and a low-quantity strategy ($p^S \approx 0.928$), both yielding an identical optimal profit of $\Pi_L^S \approx 0.517$.

The preceding equilibrium characterization sets the stage for the paper's central result. Stripped of specific functional forms, the leader's strategic advantage is bounded below by ϕ_α. Remarkably, for the vast class of log-concave demands ($\alpha = 0$), this bound distills into a global floor: the *Golden Ratio* $\phi_0 \approx 1.618$.

Proposition 1. *The equilibrium leader-to-follower profit ratio is bounded below by:*

$$\frac{\Pi_L(p^S)}{\Pi_F(p^S)} \geq \frac{1+\sqrt{5+4\alpha}}{2} \triangleq \phi_\alpha. \tag{5}$$

Proof. First, we establish the differential characterization of α-concavity. A twice-differentiable demand function is α-concave for any $\alpha \in [0, 1]$ if and only if:

$$D''(p)D(p) \leq (1-\alpha)D'(p)^2. \tag{6}$$

For $\alpha \in (0, 1]$, this bound follows from expanding the concavity condition $\frac{d^2}{dp^2}[D(p)^\alpha] \leq 0$ and dividing by the positive term $\alpha D(p)^{\alpha-2}$. For log-concavity ($\alpha = 0$), it follows from the expansion of $\frac{d^2}{dp^2}[\ln D(p)] \leq 0$.

Second, consider the leader's first-order condition. Let $p^S \in (0, p^M)$ be any Stackelberg equilibrium price. By Lemma 1, it must satisfy $D(p^S) + 3p^S D'(p^S) + (p^S)^2 D''(p^S) = 0$. Dividing this equation by $D(p^S)$ and substituting the equilibrium price elasticity

$\epsilon^S = -p^S D'(p^S)/D(p^S)$ yields:

$$1 - 3\epsilon^S + \frac{(p^S)^2 D''(p^S)}{D(p^S)} = 0. \tag{7}$$

From the differential characterization (6) of α-concavity, we have:

$$\frac{(p^S)^2 D''(p^S)}{D(p^S)} \leq (1-\alpha)\left[\frac{p^S D'(p^S)}{D(p^S)}\right]^2 = (1-\alpha)(\epsilon^S)^2. \tag{8}$$

Substituting Inequality (8) into the Equation (7) provides $1 - 3\epsilon^S + (1-\alpha)(\epsilon^S)^2 \geq 0$. Because the equilibrium elasticity $\epsilon^S \in (0,1)$, it must be bounded above by the smaller root of this quadratic polynomial:

$$\epsilon^S \leq \frac{2}{3+\sqrt{5+4\alpha}}. \tag{9}$$

This upper bound holds for all $\alpha \geq 0$, accommodating the boundary case $\alpha = 1$. Under concave demand, the quadratic inequality degenerates into a linear inequality $\epsilon^S \leq 1/3$. By Equation (4), the profit ratio is $\frac{\Pi_L(p^S)}{\Pi_F(p^S)} = \frac{1}{\epsilon^S} - 1$. As this ratio strictly decreases in ϵ^S:

$$\frac{\Pi_L(p^S)}{\Pi_F(p^S)} \geq \frac{3+\sqrt{5+4\alpha}}{2} - 1 = \frac{1+\sqrt{5+4\alpha}}{2}. \tag{10}$$

This completes the proof. ☐

On it own, the profit identity $\Pi_L/\Pi_F = 1/\epsilon^S - 1$ is silent on the phenomenon of first-mover advantage.[6] In fact, an equilibrium elasticity $\epsilon^S > 1/2$ leads to a second-mover advantage. However, the α-concave family, which includes many standard demand forms used in economics, caps the elasticity well below 1/2. This explains the persistent observation of the first-mover advantage across IO literature.

For antitrust regulators, Proposition 1 elevates the local understanding of the first-mover advantage into a global guarantee. By bounding the leader's relative market power from below, it arms authorities with a distribution-free tool to evaluate leadership dominance under demand ambiguity. Yet, while Example 3 eventually constructs a demand showing the lower bound is sharp, the leader's advantage can grow to infinity.

Example 2 (Polynomial demand)**.** Consider the following demand, parameterized by k:

$$D(p) = 1 - p^k, \quad p \in [0,1].$$

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[6] See also Gal-Or (1985), who proves that the first-mover advantage arises if and only if the players' reaction functions are downward sloping.

This demand is concave since its second derivative $D''(p) = -k(k-1)p^{k-2} \leq 0$ for all $k \geq 1$. As $D(p)$ is 1-concave, it is α-concave for any $\alpha \in [0,1]$. Under Stackelberg competition, the follower's best response quantity is $q_F(p) = -pD'(p) = kp^k$. The leader optimizes over the effective demand $q_L(p) = D(p) - q_F(p) = 1 - (k+1)p^k$, yielding the continuation profit function $\Pi_L(p) = p - (k+1)p^{k+1}$. The leader's optimality condition pins down the Stackelberg equilibrium price p^S where $(p^S)^k = \frac{1}{(k+1)^2}$. Evaluating the price elasticity of demand exactly at this equilibrium yields $\epsilon^S = \frac{1}{k+2}$. Using identity (4), the leader's relative advantage is $\frac{\Pi_L}{\Pi_F} = \frac{1}{\epsilon^S} - 1 = k+1$. Thus, α-concavity permits an unbounded profit ratio that drives the follower's share to near zero.

4 The limits of surplus sharing

At any given Stackelberg equilibrium, industry profit evaluates to $\Pi_T \triangleq p^S D(p^S)$ and deadweight loss to $\mathrm{DWL} \triangleq \int_0^{p^S} D(p)\,dp - \Pi_T$. Measuring the minimum leader's surplus and the maximum deadweight loss against total surplus or industry profit is of paramount economic importance: it quantifies the limits of market power and market distortion.

Proposition 2. *Let $\bar{\epsilon}_\alpha \triangleq \frac{2}{3+\sqrt{5+4\alpha}}$ be the cap on equilibrium elasticity from Equation (9). If the demand is α-concave for $\alpha \in (0,1]$, the leader's profit $\Pi_L \geq \mu \cdot (1+\alpha)\bar{\epsilon}_\alpha(1-\bar{\epsilon}_\alpha)(1+\alpha\bar{\epsilon}_\alpha)^{-\frac{1+\alpha}{\alpha}}$. If the demand is log-concave, $\Pi_L \geq \mu \cdot \bar{\epsilon}_0(1-\bar{\epsilon}_0)e^{-\bar{\epsilon}_0}$, where $\bar{\epsilon}_0 = \frac{3-\sqrt{5}}{2}$.*

Proof. Because demand is α-concave, $D(p)^\alpha$ is concave. For an arbitrary price $p_0 \in (0,\omega)$, the first-order Taylor expansion provides $D(p)^\alpha \leq D(p_0)^\alpha + \alpha D(p_0)^{\alpha-1}D'(p_0)(p-p_0)$ for any p. Substituting elasticity $\epsilon_{p_0} = -p_0 D'(p_0)/D(p_0)$ into the expansion and taking the $1/\alpha$-power implies that $D(p)$ is bounded above by $\overline{D}(p)$:

$$D(p) \leq \overline{D}(p) \triangleq D(p_0)\left(1 - \frac{\alpha\epsilon_{p_0}}{p_0}(p-p_0)\right)^{1/\alpha}.$$

Since $\overline{D}\left(p_0\left(1+\frac{1}{\alpha\epsilon_{p_0}}\right)\right) = 0$, the maximum willingness to pay ω must not exceed $p_0\left(1+\frac{1}{\alpha\epsilon_{p_0}}\right)$. Therefore, the area under the demand curve μ is bounded above by:

$$\mu \leq \int_0^{p_0\left(1+\frac{1}{\alpha\epsilon_{p_0}}\right)} D(p_0)\left(1 - \frac{\alpha\epsilon_{p_0}}{p_0}(p-p_0)\right)^{1/\alpha} dp = \frac{p_0 D(p_0)}{\epsilon_{p_0}(1+\alpha)}(1+\alpha\epsilon_{p_0})^{\frac{1+\alpha}{\alpha}}.$$

Rearranging this bound in terms of the leader's profit $\Pi_L(p_0) = p_0 D(p_0)(1-\epsilon_{p_0})$ yields:

$$\Pi_L(p_0) \geq \mu(1+\alpha)\left[\epsilon_{p_0}(1-\epsilon_{p_0})(1+\alpha\epsilon_{p_0})^{-\frac{1+\alpha}{\alpha}}\right]. \tag{11}$$

Let $\eta(\epsilon) \triangleq \epsilon(1-\epsilon)(1+\alpha\epsilon)^{-\frac{1+\alpha}{\alpha}}$. We have shown that for any p_0, $\Pi_L(p_0) \geq \mu(1+\alpha)\eta(\epsilon_{p_0})$.

Taking the derivative of $\eta(\epsilon)$ reveals its fundamental link to the leader's first-order condition $\eta'(\epsilon) = (1+\alpha\epsilon)^{-\frac{1+2\alpha}{\alpha}}[1-3\epsilon+(1-\alpha)\epsilon^2]$. The bracketed term is identically the quadratic derived from the leader's optimality condition. Because $\bar{\epsilon}_\alpha$ is the smaller root of this quadratic, $\eta'(\epsilon) > 0$ for $\epsilon \in (0, \bar{\epsilon}_\alpha)$ and $\eta'(\epsilon) < 0$ for $\epsilon \in (\bar{\epsilon}_\alpha, 1]$, ensuring the right-hand side of (11) is maximized at $\epsilon_{\tilde{p}} = \bar{\epsilon}_\alpha$. As any Stackelberg price p^S maximizes $\Pi_L(p)$, and there exists a price $\tilde{p} \in (0, p^M)$ such that $\epsilon_{\tilde{p}} = \bar{\epsilon}_\alpha$:

$$\Pi_L(p^S) \geq \Pi_L(\tilde{p}) \geq \mu \cdot (1+\alpha)\left[\bar{\epsilon}_\alpha(1-\bar{\epsilon}_\alpha)(1+\alpha\bar{\epsilon}_\alpha)^{-\frac{1+\alpha}{\alpha}}\right].$$

When demand is log-concave, an analogous argument applies to the concave function $\ln D(p)$, showing $D(p)$ is bounded above by $\overline{D}(p) \triangleq D(p_0)\exp\left(-\frac{\epsilon_{p_0}}{p_0}(p-p_0)\right)$ for any fixed p_0. The area under the demand curve is therefore bounded above by:

$$\mu \ \leq \ \int_0^\infty D(p_0)\exp\left(-\frac{\epsilon_{p_0}}{p_0}(p-p_0)\right)\,dp = \frac{p_0 D(p_0)}{\epsilon_{p_0}} e^{\epsilon_{p_0}}.$$

This bound leads to $\Pi_L(p_0) \geq \mu\epsilon_{p_0}(1-\epsilon_{p_0})e^{-\epsilon_{p_0}}$. Defining $\gamma(\epsilon) \triangleq \epsilon(1-\epsilon)e^{-\epsilon}$, the derivative $\gamma'(\epsilon) = e^{-\epsilon}(1-3\epsilon+\epsilon^2)$ ensures that the bounding function $\mu\gamma(\epsilon)$ strictly increases up to the root $\bar{\epsilon}_0 = \frac{3-\sqrt{5}}{2}$ and decreases thereafter. The suboptimal choice argument follows identically as before, proving $\Pi_L(p^S) \geq \mu \cdot \bar{\epsilon}_0(1-\bar{\epsilon}_0)e^{-\bar{\epsilon}_0}$. □

Proposition 2 and Example 2 suggests that there is no such lower bound for the follower profit. Because industry profit includes the leader's share, these bounds establish industry minimums. Regardless of equilibrium multiplicity, the leader—and by extension, the entire industry—is guaranteed to extract at least 25% of the potential surplus μ under concave demand, and $(\sqrt{5}-2)e^{-\frac{3-\sqrt{5}}{2}} \approx 16.1\%$ of μ under log-concave demand.

Leveraging the upper bound of the demand curve in Proposition 2, I further derive an upper bound on deadweight loss relative to total industry profit.

Proposition 3. *Let* $\bar{\epsilon}_\alpha \triangleq \frac{2}{3+\sqrt{5+4\alpha}}$ *be the cap on equilibrium elasticity from Equation* (9). *If the demand is* α*-concave for* $\alpha \in (0,1]$, $\frac{\mathrm{DWL}}{\Pi_T} \leq \frac{(1+\alpha\bar{\epsilon}_\alpha)^{\frac{1+\alpha}{\alpha}}-1}{(1+\alpha)\bar{\epsilon}_\alpha} - 1$. *If the demand is log-concave* ($\alpha = 0$), *the ratio is bounded above by* $\frac{\mathrm{DWL}}{\Pi_T} \leq \frac{e^{\bar{\epsilon}_0}-1}{\bar{\epsilon}_0} - 1$.

Proof. Let p^S be a Stackelberg price and $\epsilon^S \in (0, \bar{\epsilon}_\alpha]$ be its elasticity. Using the upper bound for the demand $\overline{D}(p)$ evaluated at $p_0 = p^S$, the deadweight loss is bounded above:

$$\mathrm{DWL} \leq \int_0^{p^S} D(p^S)\left(1-\frac{\alpha\epsilon^S}{p^S}(p-p^S)\right)^{1/\alpha} dp - \Pi_T = \left[\frac{(1+\alpha\epsilon^S)^{\frac{1+\alpha}{\alpha}}-1}{(1+\alpha)\epsilon^S} - 1\right]\Pi_T.$$

As the right-hand side increases in ϵ^S, substituting the cap $\bar{\epsilon}_\alpha$ yields the desired bound.

When the demand is log-concave ($\alpha = 0$), $D(p)$ is bounded above by $\bar{D}(p) = D(p^S)\exp\left\{-\frac{\epsilon^S}{p^S}(p - p^S)\right\}$. The deadweight loss is thus bounded above by:

$$\text{DWL} \leq \int_0^{p^S} D(p^S)\exp\left\{-\frac{\epsilon^S}{p^S}(p - p^S)\right\} dp - \Pi_T = \left[\frac{e^{\epsilon^S} - 1}{\epsilon^S} - 1\right]\Pi_T.$$

The term $\frac{e^{\epsilon^S}-1}{\epsilon^S} - 1$ increases in ϵ^S. Substituting $\bar{\epsilon}_0$ completes the proof. □

Proposition 3 suggests a relative efficiency gain from sequential competition. While a monopolist can generate deadweight loss up to $(e-2) \approx 71.8\%$ of profit under log-concave demand (Condorelli and Szentes, 2022), the Stackelberg duopoly compresses the maximum loss to 21.8% of industry profit. Increasing curvature α tightens this cap.

Finally, I demonstrate that all the bounds derived above can be achieved.

Example 3 (Exponential Demand)**.** Consider the following demand given $\alpha \in (0, 1]$:

$$D(p) = \left(1 - \frac{\alpha}{\mu(1+\alpha)}p\right)^{1/\alpha}, \ p \in \left[0, \mu\frac{1+\alpha}{\alpha}\right]. \tag{12}$$

It is straightforward to verify that the demand is α-concave since $D(p)^\alpha$ is linear. Also, $D(0) = 1$, $D\left(\mu\frac{1+\alpha}{\alpha}\right) = 0$, and D strictly decreases in p. The total surplus is μ. Under Stackelberg competition, this demand function achieves all the derived bounds. At the unique Stackelberg equilibrium, the price, the elasticity, total quantity, leader profit, follower profit, industry profit, and deadweight loss evaluate to:

$$\begin{aligned}
&p^S = \mu\frac{(1+\alpha)\bar{\epsilon}_\alpha}{1+\alpha\bar{\epsilon}_\alpha}, \quad D(p^S) = (1+\alpha\bar{\epsilon}_\alpha)^{-1/\alpha}, \quad \epsilon^S = \bar{\epsilon}_\alpha,\\
&\Pi_L = \mu(1+\alpha)\bar{\epsilon}_\alpha(1-\bar{\epsilon}_\alpha)(1+\alpha\bar{\epsilon}_\alpha)^{-\frac{1+\alpha}{\alpha}}, \quad \Pi_F = \mu(1+\alpha)\bar{\epsilon}_\alpha^2(1+\alpha\bar{\epsilon}_\alpha)^{-\frac{1+\alpha}{\alpha}},\\
&\Pi_T = \mu(1+\alpha)\bar{\epsilon}_\alpha(1+\alpha\bar{\epsilon}_\alpha)^{-\frac{1+\alpha}{\alpha}}, \quad \text{DWL} = \mu\left\{1 - (1+\alpha\bar{\epsilon}_\alpha)^{-\frac{1+\alpha}{\alpha}}\left[1 + (1+\alpha)\bar{\epsilon}_\alpha\right]\right\}.
\end{aligned}$$

When $\alpha = 0$, the demand is chosen to be $D(p) = e^{-p/\mu}$ for $p \in [0, \infty)$. It is simple to show log-concavity. Also, $D(0) = 1$, $\lim_{p\to\infty} D(p) = 0$, and D strictly decreases in p. The total surplus is μ. At the unique Stackelberg equilibrium, the price, the elasticity, total quantity, leader profit, follower profit, industry profit, and deadweight loss are given by:

$$\begin{aligned}
&p^S = \mu\bar{\epsilon}_0, \quad D(p^S) = e^{-\bar{\epsilon}_0}, \quad \epsilon^S = \bar{\epsilon}_0,\\
&\Pi_L = \mu\bar{\epsilon}_0(1-\bar{\epsilon}_0)e^{-\bar{\epsilon}_0}, \quad \Pi_F = \mu\bar{\epsilon}_0^2 e^{-\bar{\epsilon}_0},\\
&\Pi_T = \mu\bar{\epsilon}_0 e^{-\bar{\epsilon}_0}, \quad \text{DWL} = \mu\left[1 - (1+\bar{\epsilon}_0)e^{-\bar{\epsilon}_0}\right].
\end{aligned}$$

This demand function attains all the aforementioned bounds. Taking the respective ratios confirms equality across Propositions 1, 2, and 3.

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